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On a fractional boundary value problem with fractional boundary conditions

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ABSTRACT

In this paper, we consider a discrete fractional boundary value problem, for $t \in [0, b+1]_{\mathbb{N}_0}$, of the form $-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1))$, $y(\nu-2) = 0$, $[\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} = 0$, where $f : [\nu-1, \dots, \nu+b]_{\mathbb{N}_{\nu-2}} \times \mathbb{R} \to \mathbb{R}$ is continuous, $1 < \nu \leq 2$, and $0 \leq \alpha < 1$. We prove that this problem can be interpreted as a discrete multipoint problem. We also show that the problem is a generalization of some recent results. Our results provide some basic analysis of discrete fractional boundary conditions.

1. Introduction

In this paper, we consider a discrete fractional boundary value problem (FBVP) of the form

$$-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)), \quad t \in [0, b+1]_{\mathbb{N}_0}, \tag{1.1}$$

$$y(\nu-2) = 0, \tag{1.2}$$

$$[\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} = 0, \tag{1.3}$$

where $f : [\nu-1, \nu-1, \dots, \nu+b]_{\mathbb{N}_{\nu-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $1 < \nu \leq 2$, and $0 \leq \alpha < 1$, with $\alpha, \nu \in \mathbb{R}$. As we shall prove later in this paper, in case $\alpha = 0$, boundary condition (1.3) reduces to $y(\nu+b) = 0$. Thus, problem (1.1)–(1.3) can be thought of as a generalization of a discrete conjugate FBVP. The conjugate problem was studied in a paper by Atici and Eloe [1]. In particular, then, our results in this paper generalize and extend certain of the results presented in [1].

Only recently has there been an increase in the development of the theory of discrete fractional equations. In addition to [1], another recent paper by Atici and Eloe [2] developed some of the basic theory of discrete fractional IVPs. Similarly, the present author considered in [3] a right-focal discrete FBVP. Some other recent works appearing in the literature on the discrete fractional calculus include [4–7]. But so far as we are aware, there has not appeared any work in the literature on discrete FBVPs with fractional boundary conditions. In the continuous case, fractional boundary conditions have been studied in some works. For example, Li et al. [8] considered a continuous three-point FBVP with a fractional boundary condition. Similarly, a recent paper by the author [9] also considered a continuous FBVP with a fractional boundary condition. So, a natural question is what implications a fractional boundary condition has for a discrete FBVP as opposed to a continuous FBVP. It is this question which we attempt to answer in this paper. Interestingly, it should be noted that this question cannot occur in the classical case, wherein only integer-order differences are allowed. Thus, this is a question unique to the fractional calculus.

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The main contribution of this paper is to give two different interpretations of problem (1.1)–(1.3) and in particular boundary condition (1.3). More specifically, in Section 3 we show that (1.3) may be interpreted as a nonlocal condition. On the other hand, in Section 4 we derive explicitly the Green's function associated to (1.1)–(1.3). This latter approach demonstrates that our results here generalize and extend the recent paper of Atici and Eloe [1]. Finally, we point out that conspicuously we do not state explicitly any existence results for problem (1.1)–(1.3). We do this for two reasons. First, many recent results for similar problems can be easily extended to this problem without much difficulty (e.g., [3,7]), so it seems redundant to reproduce essentially such results. Second, the goal of this paper is both to elucidate the meaning of boundary condition (1.3) and to cast (1.1)–(1.3) as a generalization of [1] rather than focus on existence theory for (1.1)–(1.3).

2. Preliminaries

We first wish to collect some basic lemmas that will be important to us in the sequel. These and other related results and their proofs can be found in any of the recent papers in the literature (cf., [1,3]).

Definition 2.1. We define $t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$, for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2. The ν-th fractional sum of a function f, for $\nu > 0$, is defined to be

$$\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

for $t \in \{a+\nu, a+\nu+1, \ldots\} =: \mathbb{N}_{a+\nu}$. We also define the ν-th fractional difference for $\nu > 0$ by $\Delta^{\nu} f(t) := \Delta^N \Delta^{\nu-N} f(t)$, where $t \in \mathbb{N}_{a+\nu+N}$ and $n \in \mathbb{N}$ is chosen so that $0 \le N-1 < \nu \le N$.

Lemma 2.3. *Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then $\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}$.*

Lemma 2.4. *Let $0 \le N-1 < \nu \le N$. Then $\Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}}$, for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.*

3. Interpretation as a multipoint condition

In this section, we argue that the boundary condition (1.3) may be interpreted as a nonlocal condition.

Theorem 3.1. *Suppose that boundary conditions (1.2)–(1.3) hold. Then there exists a function $\psi : \mathbb{R}^{b+4} \to \mathbb{R}$ such that $y(\nu+b+1) = \psi(\mathbf{y})$, where we identify the vector $\mathbf{y}$ as $\mathbf{y} := (y(\nu-2), y(\nu-1), y(\nu), \ldots, y(\nu+b), y(\nu+b+1))$.*

Proof. Observe by Definition 2.2 that for $t \in [\nu-2-\alpha, \ldots, \nu+b-\alpha+1]_{\mathbb{N}_{\nu-2-\alpha}}$

$$\begin{aligned}\Delta^{\alpha} y(t) &= \Delta\Delta^{\alpha-1} y(t) = \Delta\left[\frac{1}{\Gamma(1-\alpha)} \sum_{s=\nu-2}^{t+\alpha-1} (t-s-1)^{\underline{-\alpha}} y(s)\right] \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{s=\nu-1}^{t+\alpha} (t-s)^{\underline{-\alpha}} y(s) - \frac{1}{\Gamma(1-\alpha)} \sum_{s=\nu-2}^{t+\alpha-1} (t-s-1)^{\underline{-\alpha}} y(s). \end{aligned} \tag{3.1}$$

Now, the right-hand side of (3.1) may be simplified by making the following observations. In particular, note that

$$\left[\frac{t-s}{t-s+\alpha}\right](t-s-1)^{\underline{-\alpha}} = (t-s)^{\underline{-\alpha}}, \tag{3.2}$$

which may be easily verified by Definition 2.1. Moreover, recall that boundary condition (1.2) implies that $y(\nu-2) = 0$. Consequently, we find that (3.1)–(3.2) imply that

$$\begin{aligned}&[\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} \\ &= \frac{1}{\Gamma(1-\alpha)}\left[\Gamma(1-\alpha) y(\nu+b+1)\right] + \frac{1}{\Gamma(1-\alpha)}\left[\sum_{s=\nu-1}^{t+\alpha-1}\left[1 - \frac{t-s+\alpha}{t-s}\right](t-s)^{\underline{-\alpha}} y(s)\right]_{t=\nu+b-\alpha+1}.\end{aligned} \tag{3.3}$$

So, it follows from (3.3) that

$$[\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} = y(\nu+b+1) + \frac{1}{\Gamma(1-\alpha)} \sum_{s=\nu-1}^{\nu+b}\left[1 - \frac{\nu+b-s+1}{\nu+b-\alpha-s+1}\right](\nu+b-\alpha+1-s)^{\underline{-\alpha}} y(s). \tag{3.4}$$

Let us note that (3.4) is well defined, for $\nu+b-\alpha-s+1 > 0$ for all admissible s.

Now, recall that boundary condition (1.3) implies that $[\Delta^{\alpha}y(t)]_{t=\nu+b-\alpha+1}=0$. So, from (1.3) together with (3.4) we find

$$y(\nu+b+1)=-\frac{1}{\Gamma(1-\alpha)}\sum_{s=\nu-1}^{\nu+b}\left[1-\frac{\nu+b-s+1}{\nu+b-\alpha-s+1}\right](\nu+b-\alpha+1-s)^{\underline{-\alpha}}y(s). \tag{3.5}$$

Finally, if we put $\psi(\mathbf{y})$ equal to the right-hand side of (3.5), then it is clear that the representation given in the statement of the theorem holds. $\square$

Remark 3.2. Observe that for each admissible s in (3.5), the coefficient of $y(s)$ in (3.5) is nonnegative. Indeed, we note that

$$-\frac{1}{\Gamma(1-\alpha)}\left[1-\frac{\nu+b-s+1}{\nu+b-\alpha-s+1}\right]=\frac{1}{\Gamma(1-\alpha)}\left[\frac{\nu+b-s+1}{\nu+b-\alpha-s+1}-1\right]\geq 0,$$

clearly. Since $(\nu+b-\alpha+1-s)^{\underline{-\alpha}}\geq 0$, evidently, the claim follows.

The relevance of Theorem 3.1 is that the boundary condition (1.3) may be interpreted as a nonlocal condition. In other words, problem (1.1)–(1.3) is really a specific discrete multipoint FBVP.

As a conclusion to this section, we prove two results that while not strictly necessary seem useful to know. In particular, we argue, first, that as $\alpha\to 0^{+}$, it follows that the boundary condition (1.3) reduces to $y(\nu+b+1)=0$. Second, we argue that as $\alpha\to 1^{-}$, it follows that boundary condition (1.3) reduces to $y(\nu+b+1)-y(\nu+b)=[\Delta y(t)]_{t=\nu+b}=0$. So, this shows explicitly that (1.1)–(1.3) is a continuous generalization of the problems in [1,3].

Proposition 3.3. *We find that* $\lim_{\alpha\to 0^{+}}[\Delta^{\alpha}y(t)]_{t=\nu+b-\alpha+1}=y(\nu+b+1)$.

Proof. Consider the beginning of the proof of Theorem 3.1. We found that

$$\Delta^{\alpha}y(t)=\Delta\left[\frac{1}{\Gamma(1-\alpha)}\sum_{s=\nu-2}^{t+\alpha-1}(t-s-1)^{\underline{-\alpha}}y(s)\right]. \tag{3.6}$$

By the continuity of the Gamma function as well as the fact that y lives on a discrete set, we get from (3.6) that

$$\lim_{\alpha\to 0^{+}}\Delta^{\alpha}y(t)=\Delta\left[\sum_{s=\nu-2}^{t-1}(t-s-1)^{\underline{0}}y(s)\right]=\Delta\left[\sum_{s=\nu-2}^{t-1}y(s)\right]. \tag{3.7}$$

But putting $t=\nu+b$ in (3.7) and applying the forward difference operator yields

$$\begin{aligned}\lim_{\alpha\to 0^{+}}[\Delta^{\alpha}y(t)]_{t=\nu+b-\alpha+1}&=\Delta\left[\sum_{s=\nu-2}^{t-1}y(s)\right]_{t=\nu+b+1}\\&=y(\nu+b+1)-y(\nu-2)+\sum_{j=0}^{b+1}[y(\nu+j-1)-y(\nu+j-1)]\\&=y(\nu+b+1)-y(\nu-2).\end{aligned} \tag{3.8}$$

Since $y(\nu-2)=0$, (3.8) implies that $y(\nu+b+1)=\lim_{\alpha\to 0^{+}}[\Delta^{\alpha}y(t)]$. $\square$

Proposition 3.4. *We find that* $\lim_{\alpha\to 1^{-}}[\Delta^{\alpha}y(t)]_{t=\nu+b-\alpha+1}=[\Delta y(t)]_{t=\nu+b}$.

Proof. We again begin with the proof of Theorem 3.1. In particular, we consider the right-hand side of (3.5). Now

$$\begin{aligned}&-\frac{1}{\Gamma(1-\alpha)}\sum_{s=\nu-1}^{\nu+b}\left[1-\frac{\nu+b-s+1}{\nu+b-\alpha-s+1}\right](\nu+b-\alpha+1-s)^{\underline{-\alpha}}y(s)\\&=\alpha y(\nu+b)+\sum_{s=\nu-1}^{\nu+b-1}\left[\frac{\nu+b-s+1}{\nu+b-\alpha-s+1}-1\right]\frac{(\nu+b-\alpha+1-s)^{\underline{-\alpha}}}{\Gamma(1-\alpha)}y(s).\end{aligned} \tag{3.9}$$

Observe that $\lim_{\alpha\to 1^{-}}\frac{1}{\Gamma(1-\alpha)}=0$, whereas

$$\lim_{\alpha\to 1^{-}}(\nu+b-\alpha+1-s)^{\underline{-\alpha}}=(\nu+b-s+1)^{\underline{-1}}=\frac{\Gamma(\nu+b-s+2)}{\Gamma(\nu+b-s+3)}. \tag{3.10}$$

Since $\frac{\nu+b-s+1}{\nu+b-\alpha-s+1} \to 1$ as $\alpha \to 1^-$, we conclude from (3.9)–(3.10) that

$$\lim_{\alpha\to 1^-}\left[-\frac{1}{\Gamma(1-\alpha)}\sum_{s=\nu-1}^{\nu+b}\left[1-\frac{\nu+b-s+1}{\nu+b-\alpha-s+1}\right](\nu+b-\alpha+1-s)^{\underline{-\alpha}}y(s)\right]=y(\nu+b). \tag{3.11}$$

Finally, putting (3.9) and (3.11) together implies that as $\alpha \to 1^-$ boundary condition (1.3) tends to the boundary condition $y(\nu+b+1)=y(\nu+b)$. □

4. Derivation of a Green's function

We now wish to provide a different representation of the solution to problem (1.1)–(1.3). Indeed, we have thus far viewed problem (1.1)–(1.3) by carefully examining the boundary condition (1.3) and showing that (1.3) reduces to a multipoint condition. However, a different strategy is to derive directly a Green's function for problem (1.1)–(1.3). This approach is also more congruent with the strategies in [1,3]. While the derivation of the Green's function is essentially standard, it does require careful attention to the operational properties of the discrete fractional calculus as well as the domains of the various functions; therefore, we include it in full below. We begin, however, by proving a preliminary lemma. We note that the proof of Lemma 4.1 can essentially be found in [1], though the statement of our result here is expressed in a slightly different way. So, we provide a proof in full.

Lemma 4.1. *For $\beta > 0$ and all $\mu \in \mathbb{R}$ for which the following is defined, we find that*

$$\Delta^{\beta}t^{\underline{\mu}}=\frac{\Gamma(\mu+1)t^{\underline{\mu-\beta}}}{\Gamma(\mu-\beta+1)}. \tag{4.1}$$

Proof. Let N be the unique positive integer such that $0 \le N-1 < \beta \le N$. Using the known formula for $\Delta^{-\nu}t^{\underline{\mu}}$ (see [1, Lemma 2.1]), where $\nu > 0$, we find that

$$\begin{aligned}\Delta^{\beta}t^{\underline{\mu}}&=\Delta^{N}\Delta^{-(N-\beta)}t^{\underline{\mu}}=\Delta^{N}\left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+N-\beta+1)}t^{\underline{\mu+N-\beta}}\right]\\&=\frac{\Gamma(\mu+1)(\mu+N-\beta)\cdots(\mu-\beta+1)t^{\underline{\mu-\beta}}}{(\mu+N-\beta)\cdots(\mu-\beta+1)\Gamma(\mu-\beta+1)}=\frac{\Gamma(\mu+1)t^{\underline{\mu-\beta}}}{\Gamma(\mu-\beta+1)},\end{aligned} \tag{4.2}$$

which shows that (4.1) holds, and so, completes the proof. □

Remark 4.2. One can compare the result of Lemma 4.1 to the corresponding result in the monograph on the continuous fractional calculus by Podlubny [10, Section 2.3.4]. One finds that Lemma 4.1 is precisely what is expected to be true.

For use in the sequel, we introduce the following notation.

$$T_1 := \left\{(t,s)\in[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}\times[0,b+1]_{\mathbb{N}_0} : 0\le s<t-\nu+1\le b+1\right\}$$
$$T_2 := \left\{(t,s)\in[\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}\times[0,b+1]_{\mathbb{N}_0} : 0\le t-\nu+1\le s\le b+1\right\}.$$

Theorem 4.3. *Let $h : [\nu-1,\nu+b]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$ be given. Then the unique solution to the problem*

$$-\Delta^{\nu}y(t)=h(t+\nu-1),\quad t\in[0,b+1]_{\mathbb{N}_0}, \tag{4.3}$$
$$y(\nu-2)=0, \tag{4.4}$$
$$[\Delta^{\alpha}y(t)]_{t=\nu+b-\alpha+1}=0, \tag{4.5}$$

is $\sum_{s=0}^{b+1}G(t,s)h(s+\nu-1)$, where $G : [\nu-2,\nu+b+1]_{\mathbb{N}_{\nu-2}}\times[0,b+1]_{\mathbb{N}_0}$, defined by

$$G(t,s):=\begin{cases}\dfrac{t^{\underline{\nu-1}}(\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}}}{(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}\Gamma(\nu)}-\dfrac{(t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}, & (t,s)\in T_1\\[2ex] \dfrac{t^{\underline{\nu-1}}(\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}}}{(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}\Gamma(\nu)}, & (t,s)\in T_2,\end{cases} \tag{4.6}$$

is the Green's function for the problem.

Proof. We know that the general solution to problem (4.3) is given by

$$y(t)=C_1t^{\underline{\nu-1}}+C_2t^{\underline{\nu-2}}-\Delta^{-\nu}h(t), \tag{4.7}$$

where C_1 and C_2 are real-valued constants to be determined. Boundary condition (4.4) implies at once that $C_2 = 0$. On the other hand, applying boundary condition (4.5)–(4.7) implies that

$$\begin{aligned} 0 &= [\Delta^{\alpha} y(t)]_{t=\nu+b-\alpha+1} \\ &= C_1 \left[\Delta^{\alpha} t^{\underline{\nu-1}}\right]_{t=\nu+b-\alpha+1} - \left[\Delta^{\alpha-\nu} h(t)\right]_{t=\nu+b-\alpha+1}, \end{aligned} \tag{4.8}$$

where we have used the fact that $\Delta^{\alpha}\Delta^{-\nu} y(t) = \Delta^{\alpha-\nu} y(t)$, which just follows from the observation that

$$\Delta^{\alpha}\Delta^{-\nu} y(t) = \Delta^{N}\Delta^{\alpha-N}\Delta^{-\nu} y(t) = \Delta^{N}\Delta^{\alpha-N-\nu} y(t) = \Delta^{N+\alpha-N-\nu} y(t) = \Delta^{\alpha-\nu} y(t), \tag{4.9}$$

where $N \in \mathbb{N}$ is the unique number satisfying $0 \le N-1 \le \alpha \le N$. (That $\Delta^{N}\Delta^{\alpha-N-\nu} y(t) = \Delta^{N+\alpha-N-\nu} y(t)$ follows from [2, Theorem 2.3].) We have also used the well-known fact that fractional sums commute; see [5, Theorem 2.2].

Now, let us observe that by Lemma 4.1

$$\Delta^{\alpha} t^{\underline{\nu-1}} = \frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)} t^{\underline{\nu-\alpha-1}}. \tag{4.10}$$

Moreover, we find that

$$\begin{aligned} -\left[\Delta^{\alpha-\nu} h(t)\right]_{t=\nu+b-\alpha+1} &= \left[\frac{-1}{\Gamma(\nu-\alpha)} \sum_{s=0}^{t+\alpha-\nu} (t-s-1)^{\underline{\nu-\alpha-1}} h(s+\nu-1)\right]_{t=\nu+b-\alpha+1} \\ &= -\frac{1}{\Gamma(\nu-\alpha)} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1). \end{aligned} \tag{4.11}$$

So, putting (4.8), (4.10) and (4.11) together imply that

$$C_1 = \frac{1}{(\nu+b-\alpha+1)^{\underline{\nu-\alpha-1}}\Gamma(\nu)} \sum_{s=0}^{b+1} (\nu+b-\alpha-s)^{\underline{\nu-\alpha-1}} h(s+\nu-1). \tag{4.12}$$

Finally, then, using both (4.10) and (4.12) to rewrite (4.8), we find that the conclusion of this theorem holds. □

Remark 4.4. It is easy to verify that in case $\alpha = 0$, (4.6) reduces to the Green's function derived in [1]. Moreover, as $\alpha \to 1^{-}$, we find that (4.6) tends to the Green's function derived in [3]. In particular, then, Theorem 4.3 once again asserts that problem (1.1)–(1.3) is really a generalization of the problems considered in both [1,3].

Remark 4.5. Those readers familiar with some of the recent literature in the theory of continuous FBVPs will recognize that (4.6) has a relatively expected form; see, for example, a recent paper by the author [9].

5. Conclusions

In this paper, we have provided two different interpretations for boundary condition (1.3). First, we have shown that (1.1)–(1.3) is really a multipoint discrete FBVP in disguise. Second, we have shown by way of an explicit Green's function that (1.1)–(1.3) continuously generalizes the conjugate FBVP considered in [1]. It seems clear that considerably more work on problem (1.1)–(1.3) could be completed. For example, in this paper we did not derive any of the properties of $G(t, s)$. In the discrete fractional case, somewhat in contrast to the continuous case, verification of these properties can be quite delicate (e.g., [3,7]), and so, this analysis might very well be mathematically interesting and nontrivial. Furthermore, it might be of interest to see what happens in the case of more complicated boundary conditions such as multiple fractional boundary conditions or higher-order problems (i.e., when $\nu > 2$ such as might occur in, say, the $(n-1, 1)$ problem). Succinctly, there seem to be considerable possibilities for future work to address more complicated interplay between fractional boundary conditions and discrete FBVPs.

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